Exam Symmetry in Physics

Date	April 5, 2016
Room	V5161.0165 & V 5161.0041b
Time	14:00 - 17:00
Lecturer	D. Boer

- Write your name and student number on every separate sheet of paper
- You are not allowed to use the lecture notes, nor other notes or books
- The weights of the exercises are given in the table below
- Answers may be given in Dutch
- Illegible handwriting will be graded as incorrect
- Good luck!

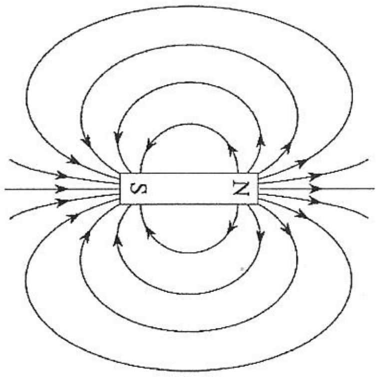
Weighting

1a)	7	2a)	8	3a)	4
1b)	7	2b)	4	3b)	6
1c)	4	2c)	8	3c)	4
1d)	4	2d)	4	3d)	6
1e)	7	2e)	6	3e)	5
1f)	6				

$$\text{Result} = \frac{\sum \text{points}}{10} + 1$$

Exercise 1

Consider a bar magnet as in the figure.



Take the central axis through the north and south pole (indicated by N and S) to be the z axis and call the two-dimensional cross section of the bar magnet orthogonal to this central axis the base.

(a) Consider the base to be a square. Identify all symmetry transformations that leave the bar magnet (without N and S written on it) *and* its magnetic field invariant. These symmetries form the group C_{4h}. Hint: there is only one reflection plane.

(b) Argue, using geometric arguments, that none of the symmetry operations are conjugated to each other (hence only to themselves) and conclude whether or not the symmetry group is Abelian.

To simplify the analysis, from now on assume the base to be a non-square rectangle, such that the symmetry group is reduced to C_{2h}.

(c) Give an identification between elements of C_{2h} and D_2 and argue that the two groups are isomorphic (as opposed to C_{4h} and D_4).

(d) Construct the character table of C_{2h} and explain how the entries are obtained.

(e) Give the three-dimensional vector and axial-vector representations, called D^V and D^A, for the two transformations that generate C_{2h}, and check whether the determinants are as expected for rotations and reflections of (axial-)vectors.

(f) Decompose D^V and D^A of C_{2h} into irreps and show that the answers are in agreement with a bar magnet having a permanent magnetic dipole moment.

Exercise 2

Consider a two-dimensional electron system as in studies of the Quantum Hall effect. Here a magnetic field is pointing in the z direction orthogonal to the plane in which the electrons move. Electric fields $\vec{E}$ and electric currents $\vec{J}$ in that two-dimensional plane are related by a so-called resistivity tensor ρ, according to $E_i = \rho_{ij} J_j$ $(i, j = 1, 2)$. Here and below summation over repeated indices is implicit.

(a) Use the transformation properties of the equation $E_i = \rho_{ij} J_j$ to derive that ρ_{ij} transforms into $\rho'_{kl} = D^V_{ki} D^V_{lj} \rho_{ij}$ under (subgroups of) rotations.

(b) Show that ρ_{ij} is invariant if it satisfies $\rho D^V = D^V \rho$.

(c) Assume the system is invariant under rotations around the z axis. Explain why in the case of $SO(2)$ symmetry, $\rho_{11} = \rho_{22}$ and $\rho_{12} = -\rho_{21}$.

(d) Explain why the trace of ρ_{ij} transforms as a scalar.

(e) Explain why $\epsilon_{ij}\rho_{ji}$ transforms as a pseudo-scalar and why it can depend linearly on the magnetic field (as it does in the case of the Quantum Hall effect). Here ϵ_{ij} is the two-dimensional analogue of ϵ_{ijk}: $\epsilon_{11} = \epsilon_{22} = 0$ and $\epsilon_{12} = -\epsilon_{21} = 1$. (Hint: consider how ρ transforms under an explicit two-dimensional reflection.)

Exercise 3

Consider the group $SU(2)$ of unitary 2×2 matrices with determinant equal to 1. Consider its action on the angular momentum states $|s, m_s\rangle$ through the operator

$$U(\theta, \hat{n}) = \exp\left(-\frac{i}{\hbar}\theta\, \hat{n} \cdot \vec{S}\right).$$

(a) Write down the explicit matrix for S_z acting on the space of $|\frac{1}{2}, m_s\rangle$ states.

(b) Write down the explicit matrix representation $D^{(s=\frac{1}{2})}$ of $U(\theta, \hat{z})$, i.e. for $\hat{n} = \hat{z}$.

(c) Show that $D^{(s=\frac{1}{2})} \in SU(2)$ and determine the range of θ.

(d) Show that this set of matrices do not form a (complex) representation of $SO(2)$, but rather a projective representation.

(e) Use the character of the $s = \frac{1}{2}$ representation to show that it is equivalent, but not equal to its complex conjugate representation.